ON FORCED OSCILLATIONS AND STABILITY OF QUASI-HARMONIC SYSTEMS

(O VYNUZHDENNYKH KOLEBANIIAKH I USTOICHIVOSTI KVAZIGARMONICHESKIKH SISTEM)

PMM Vol.25, No.2, 1961, pp. 284-293

E. N. ROZENVASSER
(Leningrad)

(Received December 23, 1960)

In the present work there is presented a method for investigating forced oscillations and the stability of systems described by linear differential equations with periodic coefficients. The method is based on the reduction of the problem to the solution of a Fredholm integral equation of the second kind.

1. Derivation of an auxiliary formula. Let us consider the system of differential equations

$$\dot{x}_k = \sum_{\alpha=1}^{n} b_{k\alpha} x_\alpha + F_k(t) \qquad (k = 1, \ldots, n) \tag{1.1}$$

where the $b_{k\alpha}$ are constants and the $F_k(t)$ are given functions of time such that

$$F_k(t) = F_k(t + T) \tag{1.2}$$

The functions F_k can be assumed to be piece-wise continuous. For what follows it is sufficient to assume that the characteristic equation

$$D(\lambda) = |b_{k\alpha} - \lambda\delta_{k\alpha}| = 0 \qquad (\delta_{k\alpha} \text{ is Kronecker's symbol}) \tag{1.3}$$

does not have roots $\lambda = \lambda_\rho$ $(\rho = 1, \ldots, n)$ which are equal

$$\psi_s = \frac{2\pi i}{T} s, \qquad i = \sqrt{-1}$$

where s is zero or any integer.

It is required to find a periodic solution of period T of the system

(1.1). This problem has been studied in detail in [1]. Below, there is given a new form of the solution which is needed in the sequel. Let

$$x_s = x_s(t) \qquad (s = 1, \ldots, n) \tag{1.4}$$

be the sought periodic solution of (1.1).

In [1] it is shown that

$$x_s(t) = \sum_{\rho=1}^{n} \frac{e^{\lambda_\rho t}}{1 - e^{-\lambda_\rho T}} \sum_{j=1}^{n} \frac{\Delta_{js}(\lambda_\rho)}{D'(\lambda_\rho)} \left[\int_0^T e^{-\lambda_\rho \tau} F_j(\tau)\, d\tau - \right.$$
$$\left. - (1 - e^{-\lambda_\rho T}) \int_0^t e^{-\lambda_\rho \tau} F_j(\tau)\, d\tau \right] \qquad (s = 1, \ldots, n) \tag{1.5}$$

where $\Delta_{js}(\lambda)$ is the algebraic cofactor of the element of the determinant (1.3) whish stands in the jth row and sth column, and it is assumed that no two of the numbers λ_ρ $(\rho = 1, \ldots, n)$ are equal.

From (1.5) it follows directly that

$$x_s(0) = \sum_{k=1}^{n} \int_0^T u_{sk}(z) F_k(z)\, dz \tag{1.6}$$

where

$$u_{sk}(z) = \sum_{\rho=1}^{n} \frac{\Delta_{sk}(\lambda_\rho)}{D'(\lambda_\rho)} \frac{e^{-\lambda_\rho \tau}}{1 - e^{-\lambda_\rho \tau}} \tag{1.7}$$

Let us now suppose that in place of the functions (1.2) we have the functions

$$F_k^\tau(t) = F_k(t + \tau)$$

on the right-hand side of Equation (1.1), where τ is a parameter. The corresponding periodic solution will be denoted by

$$x_s = x_s^\tau(t)$$

Making use of (1.6), we obtain at once

$$x_s^\tau(0) = \sum_{k=1}^{n} \int_0^T u_{sk}(z) F_k(z + \tau)\, dz \qquad (s = 1, 2, \ldots, n) \tag{1.8}$$

On the other hand, having made a change of the variable $y = t + \tau$ in (1.1), we obtain

$$x_s(t+\tau) = x_s^{\tau}(t)$$

From this we find that

$$x_s(\tau) = x_s^{\tau}(0) \tag{1.9}$$

Substituting this expression into (1.8) and replacing τ by t, we obtain an integral representation of the solution (1.4):

$$x_s(t) = \sum_{k=1}^{n} \int_0^T u_{sk}(z) F_k(z+t)\, dz \qquad (s=1, 2, \ldots, n) \tag{1.10}$$

Next, we note that Equations (1.10) remain valid if in place of U_{sk} we use the formulas

$$\Phi_{sk}(z) = \Phi_{sk}(z+T), \qquad \Phi_{sk}(z) = u_{sk}(z) \quad \text{for } 0 < z < T \tag{1.11}$$

i.e. one may write

$$x_s(t) = \sum_{k=1}^{n} \int_0^T \Phi_{sk}(z) F_k(z+t)\, dz \qquad (s=1, 2, \ldots, n) \tag{1.12}$$

Through the introduction of a new variable, Expression (1.12) can be transformed into the form

$$x_s(t) = \sum_{k=1}^{n} \int_t^{t+T} \Psi_{sk}(t-y) F_k(y)\, dy \qquad (s=1, 2, \ldots, n) \tag{1.13}$$

where

$$\Psi_{sk}(z) = \Phi_{sk}(-z) \tag{1.14}$$

The Fourier expansion of the function Ψ_{sk} has the form

$$\Psi_{sk}(z) = -\frac{1}{T} \sum_{m=-\infty}^{\infty} \frac{\Delta_{sk}(\psi_m)}{D(\psi_m)} e^{\psi_m z} \tag{1.15}$$

Formula (1.15) follows from (1.14) and the equation

$$\frac{1}{T}\int_0^T u_{sk}(z) e^{-\psi_m z}\, dz = \frac{1}{T}\sum_{\rho=1}^{n} \frac{\Delta_{sk}(\lambda_\rho)}{D'(\lambda_\rho)} \frac{1}{\lambda_\rho + \psi_k} = -\frac{1}{T}\frac{\Delta_{sk}(-\psi_m)}{D(-\psi_m)}.$$

Noting, furthermore, that in accordance with (1.15) the function $\Psi_{sk}(t-\tau)$ is a periodic function of period T in each of its arguments, and taking into account a known property of the integral of a periodic function, one may consider the limits of integration in (1.13) as constants equal to 0 and T, i.e.

$$x_s(t) = \sum_{k=1}^{n} \int_0^T \Psi_{sk}(t-y) F_k(y)\, dy \tag{1.16}$$

We also note some other properties of the function Ψ_{sk}. From (1.14) and (1.17) we obtain

$$\Psi_{sk}(z) = u_{sk}(T-z) = \sum_{\rho=1}^{n} \frac{\Delta_{sk}(\lambda_\rho)}{D'(\lambda_\rho)} \frac{e^{\lambda_\rho z}}{e^{\lambda_\rho T}-1} \qquad \text{for } 0<z<T \tag{1.17}$$

The following relation also holds:

$$\Psi_{sk}(0) = \frac{1}{2} \sum_{\rho=1}^{n} \frac{\Delta_{sk}(\lambda_\rho)}{D'(\lambda_\rho)} \coth \frac{\lambda_\rho T}{2} \tag{1.18}$$

This follows from the property of Fourier series

$$\Psi_{sk}(0) = \Psi_{sk}(T) = \frac{u_{sk}(0) + u_{sk}(T)}{2}$$

One can show also that the function $\Psi_{sk}(t-\tau)$ is continuous in the square

$$0<\tau<T, \qquad 0<t<T$$

if $k \neq s$, while the functions Ψ_{sk} have discontinuities equal to 1 along the diagonal $t=\tau$.

All this was considered under the assumption that Equation (1.3) does not have multiple roots. It is, however, not difficult to show that Formulas (1.16) and (1.15) are valid without this assumption.

2. Forced oscillations of a quasi-harmonic system. We shall consider the system of differential equations

$$x_k = \sum_{\alpha=1}^{n} b_{k\alpha} x_\alpha + \mu \sum_{\alpha=1}^{n} m_{k\alpha}(t) x_\alpha + F_k(t) \qquad (k=1,\ldots,n) \tag{2.1}$$

where the $b_{k\alpha}$ are the same as in (1.1), $m_{k\alpha}(t)$ and $F_k(t)$ are given functions of time of period T, and μ is some parameter.

We shall seek a periodic solution of the system (2.1) which is of period T. Let us suppose that such a solution of (1.4) exists. Then, taking into account (1.6), we find that this solution must satisfy the system of Fredholm integral equations of the second kind:

$$x_s(t) = \mu \sum_{k=1}^{n} \int_0^T \sum_{\alpha=1}^{n} \Psi_{sk}(t-y)\, m_{k\alpha}(y)\, x_\alpha(y)\, dy + \int_0^T \Psi_{sk}(t-y)\, F_k(y)\, dy$$

$$(s = 1, \ldots, n) \tag{2.2}$$

Making use of a well-known procedure for the reduction of a system of integral equations to a single integral equation, and taking into account the periodicity of the functions occurring under the integral sign, we reduce the system of equations (2.2) to one equation:

$$X(t) = X(t+nT), \qquad X(t) = x_s(t) \quad \text{when } (s-1)T < t < sT$$

$$F(t) = F(t+nT), \qquad F(t) = \int_0^T \sum_{k=1}^{n} \Psi_{sk}(t-y)\, F_k(y)\, dy \tag{2.3}$$

$$\text{when } (s-1)T < t < sT$$

Let us introduce a kernel, defined in the square

$$0 < t < nT, \qquad 0 < y < nT \tag{2.4}$$

by means of the formulas

$$K_1(t, y) = \sum_{k=1}^{n} \Psi_{sk}(t-y)\, m_{k\alpha}(y) = K_{s\alpha}(t, y)$$

when

$$(s-1) < T < t < sT, \qquad (\alpha - 1)T < y < \alpha' T \tag{2.5}$$

Making use of the introduced notation, we may write the system (2.2) as one equation of the form

$$X(t) = \mu \int_0^{nT} K_1(t, y)\, X(y)\, dy + F(t) \tag{2.6}$$

Let us note that the homogeneous equation

$$X(t) = \mu \int_0^{nT} K_1(t, y)\, X(y)\, dy \tag{2.7}$$

will have non-zero solutions if, and only if, the homogeneous system corresponding to the system (2.1) has a periodic solution of period T. In view of this it is not difficult to show that the known theorems on the existence of a periodic solution of the system (2.1), which are given in [2], are simple rephrasings of the known theorems of Fredholm [3] applied to Equation (2.6).

Let us now proceed with the solution of Equation (2.6). In the general case

$$X(t) = \mu \int_0^{nT} R(\mu,t,y) F(y)\, dy + F(t) \tag{2.8}$$

where $R(\mu, t, y)$ is the resolvent of the kernel in (2.5). For small enough $|\mu|$, $R(\mu, t, y)$ is a power series in μ:

$$R(\mu,t,y) = \sum_{i=0}^{\infty} K_{i+1}(t,y)\, \mu^i \tag{2.9}$$

where K_i are the iterated kernels evaluated by the recurrence formula

$$K_{i+1}(t,y) = \int_0^{nT} k_1(t, z)\, k_i(z, y)\, dz \tag{2.10}$$

One can show that the sequence of operations associated with the representation of the series (2.9) and of the solution (2.8) is entirely equivalent to the determination of the periodic solution of the system (2.1) in the "non-resonance case" [2] by the method of a small parameter. From the theory of Fredholm's equation it is known that the series (2.9) converges for all μ such that

$$0 < |\mu| < |\mu_0| \tag{2.11}$$

where μ_0 is the smallest (in absolute value) μ for which the homogeneous equation (2.7) has a non-zero solution, and that the series diverges when $|\mu| > |\mu_0|$. On the other hand, from what was said above, it follows that μ_0 is the smallest (in absolute value) value of the parameter for which the homogeneous system corresponding to (2.1) will have a periodic solution. From this it follows directly that if $|\mu| \geqslant |\mu_0|$ it is impossible to construct a periodic solution of the system (2.1) by the method of a small parameter. For many problems it is possible to estimate μ_0 and to have a method for constructing forced oscillations for arbitrary values of the parameter. These results can be obtained by applying in the solution of Equation (2.6) Fredholm's method which gives the solution for arbitrary values of the parameter. In the general case, the resolvent is a meromorphic function of the parameter μ:

$$R(\mu, t, y) = \frac{\Delta(\mu, t, y)}{\Delta(\mu)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu^n \Delta_n(t, y) \Big/ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu^n \Delta_n \tag{2.12}$$

From this it follows that in the general case the periodic solution of the system (2.1) is a meromorphic function of the parameter. Expression (2.9) is the Taylor series expansion of the function (2.12). Therefore, the number of μ_0 can be determined as the smallest (in absolute value) root of the equation

$$\Delta(\mu_0) = 0$$

For practical evaluations of the coefficients entering in the numerator and denominator of (2.12), one can use the recurrence formulas [4]

$$\Delta_n(t,y) = K_1(t,y) - n \int_0^{nT} K_1(t,z)\,\Delta_{n-1}(z,y)\,dz$$

$$\Delta_{n+1} = \int_0^{nT} \Delta_n(t,t)\,dt, \qquad \Delta_0(t,y) = K_1(t,y), \qquad \Delta_0 = 1 \tag{2.13}$$

It is not difficult to establish that the coefficient $\Delta_n(t, y)$ is a linear combination of the iterated kernels $K_1(t, y), \ldots, K_n(t, y)$. For their practical computation one can use the following procedure.

We introduce the matrix $\|K_1(t, y)\|$ which consists of the functions $K_{s\alpha}(t, y)$. We shall evaluate the iterated matrices by means of the formula

$$\|K_i(t,y)\| = \int_0^T \|K_i(t,z)\|\ \|K_{i-1}(z,y)\|\,dy \tag{2.14}$$

The corresponding elements of the matrix $\|K_i(t, y)\|$ will be denoted by $K_{s\alpha}{}^i(t, y)$. With each matrix $\|K_i(t, y)\|$ we associate a scalar function of two arguments defined in the square (2.4) by the formula

$$K_i(t,y) = K_{s\alpha}^i \quad \text{when } (s-1)T < t < sT, \ (\alpha-1)T < y < \alpha T \tag{2.15}$$

By direct evaluation one can verify that the function $K_i(t, y)$ thus constructed coincides with the ith iterated kernel (2.10). In the evaluation of the coefficients Δ_n it is necessary to evaluate integrals of the form

$$\alpha_i = \int_0^{nT} K_i(t,t)\,dt \tag{2.16}$$

Taking into account (2.15) and the periodicity of all iterated kernels in each of their arguments, we find that

$$\alpha_i = \int_0^T \mathrm{sp}\,\| K_i(t, t)\|\, dt \qquad \left(\mathrm{sp}\,\| K_i(t, y)\| = \sum_{\alpha=1}^{n} K_{\alpha\alpha}{}^{i}(t, y)\right) \quad (2.17)$$

One can also construct the solution of Equation (2.6) in a different order by making use of known formulas for the coefficients of Fredholm [4] series.

As an example, let us consider the case of system (2.1):

$$\dot{x}_k = \sum_{\alpha=1}^{n} b_{k\alpha} x_\alpha + \mu h_k f(t)\sigma + F_k(t) \qquad (k = 1, 2, \ldots, n) \qquad (2.18)$$

$$\sigma = \sum_{s=1}^{n} j_s x_s, \qquad f(t) = \sum_{k=-\infty}^{\infty} f_k e^{\psi_k t}$$

Making use of (2.2) we obtain

$$x_s(t) = \mu \int_0^T \Psi_s(t-y) f(y)\,\sigma(y)\, dy + \sum_{k=1}^{n} \int_0^T \Psi_{sk}(t-y) F_k(y)$$

where

$$\Psi_s(t-y) = -\frac{1}{T} \sum_{m=-\infty}^{\infty} \sum_{k=1}^{n} \frac{h_k \Delta_{sk}(\psi_m)}{D(\psi_m)} e^{\psi_m (t-y)} = \frac{1}{T} \sum_{m=-\infty}^{\infty} \frac{H_s(\psi_m)}{D(\psi_m)} e^{\psi_m (t-y)}$$

Hence, making use of the expression σ from (2.18), we obtain

$$\sigma(t) = \mu \int_0^T \Psi(t-y) f(y)\,\sigma(y)\, dy + s(t) \qquad (2.19)$$

where

$$\Psi(t-y) = -\frac{1}{T} \sum_{p=1}^{n} \sum_{k=1}^{n} \sum_{m=-\infty}^{\infty} \frac{h_{pjk}\,\Delta_{pk}(\psi_m)}{D(\psi_m)} e^{\psi_m (t-y)} = \frac{1}{T} \sum_{m=-\infty}^{\infty} \frac{M(\psi_m)}{D(\psi_m)} e^{\psi_m (t-y)}$$

$$s(t) = \int_0^T \sum_{s=1}^{n} \sum_{k=1}^{n} j_s\, \Psi_{sk}(t-y) F_k(y)\, dy$$

Utilizing the known formulas (2.13), we shall construct the resolvent $R(t, y)$ of the kernel

$$\Psi(t-y) f(y) \tag{2.20}$$

After evaluation with an accuracy of up to μ^3, we obtain

$$R(t, y) = f(y) \frac{L_1 - \mu(L_1\Delta_1 - L_2) + {}^1/_2\mu^2(L_1\Delta_2 - 2L_2\Delta_1 + 2L_3)}{1 - \mu\Delta_1 + {}^1/_2\mu^2\Delta_2}$$

$$L_1(t, y) = \sum_{k=-\infty}^{\infty} \frac{M(\psi_k)}{D(\psi_k)} e^{\psi_k(t-y)} \tag{2.21}$$

$$L_2(t, y) = \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} \frac{M(\psi_k)}{D(\psi_k)} f_{k-r} \frac{M(\psi_r)}{D(\psi_r)} e^{\psi_k t - \psi_r y}$$

$$L_3(t, y) = \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \frac{M(\psi_k)}{D(\psi_k)} f_{k-r} \frac{M(\psi_r)}{D(\psi_r)} f_{r-p} \frac{M(\psi_p)}{D(\psi_p)} e^{\psi_k t - \psi_p y}$$

$$\Delta_1 = f_0 \sum_{k=-\infty}^{\infty} \frac{M(\psi_k)}{D(\psi_k)} \qquad \text{etc.}$$

Everything that has been said in this section can be extended directly to the case of finding almost periodic solutions of the system (2.1) if the functions $F_k(t)$ have the form

$$F_k(t) = e^{i\nu t} \varphi_k(t)$$

where the functions $\phi_k(t)$ are of period T, and ν is a real number. As is known [2], if in the case considered there exists an almost periodic solution, then it will be of the form

$$x_s(t) = e^{i\nu t} \chi_s(t) \qquad (s = 1, 2, \ldots, n)$$

where $\chi_s(t)$ is of period T. Hence, if one introduces new unknowns

$$y_s = x_s e^{-i\nu t}$$

then the system (2.1) will take on the form

$$\dot{y}_k = \sum_{\alpha=1}^{n} (b_{k\alpha} + i\nu\delta_{k\alpha}) x_\alpha + \mu \sum_{\alpha=1}^{n} m_{k\alpha}(t) x_\alpha + \varphi_k(t) \tag{2.22}$$

and the problem will have been reduced to finding periodic solutions of period T of the system (2.2).

3. Stability of a quasi-harmonic system. Problems on the investigation of the parametric resonance, of the stability of periodic motions, and other problems lead to the necessity of examining the properties of homogeneous systems of linear differential equations with periodic coefficients. Let us consider the homogeneous system which corresponds to (3.1):

$$\dot{x}_k = \sum_{\alpha=1}^{n} b_{k\alpha}x_\alpha + \mu \sum_{\alpha=1}^{n} m_{k\alpha}(t)\, x_\alpha \qquad (k = 1, 2, \ldots, n) \tag{3.1}$$

In accordance with the theory of Floquet [2], we shall look for solutions of the form

$$x_s(t) = e^{\lambda_i t} f_{is}(t) \qquad (s = 1, 2, \ldots, n) \tag{3.2}$$

where the $f_{is}(t)$ are periodic functions of period T. By making the substitution

$$x_s(t) = e^{\lambda t} y_s(t)$$

we obtain the system

$$\dot{y}_k = \sum_{\alpha=1}^{n} b_{k\alpha}x_\alpha - \lambda\delta_{k\alpha}x_\alpha + \mu \sum_{\alpha=1}^{n} m_{k\alpha}(t)\, y_\alpha \tag{3.3}$$

The characteristic exponents λ_i have to be determined from the condition that for $\lambda = \lambda_i$ the system (3.3) will have a periodic solution of period T. But if there is no such solution, then in view of what has been said above the system of integral equations

$$X(t) = \mu \sum_{k=1}^{n} \sum_{\alpha=1}^{n} \int_0^T \Psi_{s\alpha}(\lambda, t - y)\, m_{\alpha k}(y)\, x_k(y)\, dy \tag{3.4}$$

where

$$\Psi_{sk}(\lambda, z) = -\frac{1}{T} \sum_{m=-\infty}^{\infty} \frac{\Delta_{sk}(\lambda + \psi_m)}{D(\lambda + \psi_m)} e^{\psi_m z} \tag{3.5}$$

will have a non-trivial solution.

Just as above, the system (3.4) can be reduced to a single integral equation with a kernel defined in the square (2.4) and depending on the parameter λ. Then, a necessary and sufficient condition for the existence of a periodic solution of that equation will be the vanishing of the Fredholm determinant, which in the given case is a function of λ:

$$\Delta(\lambda_i, \mu) = 0 \tag{3.6}$$

Equation (3.6) can be used for the determination of the characteristic exponents, and also for the separation of the regions of stability in the space of the parameters. In addition to the ways indicated in the preceding section for the evaluation of $\Delta(\lambda, \mu)$, one can utilize the relation

$$-\frac{1}{\Delta(\lambda, \mu)} \frac{d\Delta(\lambda, \mu)}{d\mu} = \int_0^{nT} R(\lambda, \mu, t, t)\, dt \tag{3.7}$$

By substituting on the right-hand side of (3.7) for $\Delta(\mu)$ an infinite series, and taking into account (2.9) and (2.16), we obtain

$$\left(\sum_{n=1}^{\infty} \mu^{n-1}\alpha_n\right)\left(1 + \sum_{n=1}^{\infty} \frac{(-\mu)^n}{n!}\Delta_n\right) = \sum_{n=1}^{\infty} \frac{(-\mu)^{n-1}}{(n-1)!}\Delta_n$$

Hence, equating coefficients of equal powers of μ, we determine successively the coefficients Δ_i:

$$\Delta_1 = \alpha_1, \qquad \Delta_2 = \alpha_1^2 - \alpha_2, \qquad \Delta_3 = \alpha_1^3 - 3\alpha_1\alpha_2 + 2\alpha_3 \qquad \text{etc.} \tag{3.8}$$

where, in accordance with (3.5), the quantity a_i is a function of λ. If one looks for a solution of the system (3.3) or (3.4) in the form of a trigonometric series

$$y_i(t) = \sum_{k=-\infty}^{\infty} u_{ik} e^{\psi_k t}$$

then one can determine the coefficients U_{ik} by means of an infinite system of linear equations. The determinant of this system which is a function of λ and μ is called Hill's determinant of the system (3.4).

One can show that the expansion of the determinant in powers of μ coincides with the expression for $\Delta(\lambda, \mu)$, where the coefficients $\Delta_i(\lambda)$ are determined in accordance with (3.8). In other words, Hill's determinant is the denominator of the resolvent kernel of the system (3.4). It should be noted that the choice of the constant matrix $\|B\|$, constructed from the numbers $b_{k\alpha}$, is essentially arbitrary and corresponds to various choices of the parameter. In particular, one may select for the matrix $\|B\|$ in (3.3) the matrix ΛE (E is the unit matrix). Then the corresponding system of integral equations will take on the form

$$x_s(t) = \int_0^T \overline{\Phi}(t-\tau) \sum_{\alpha=1}^{n} (b_{s\alpha} + \mu m_{s\alpha}(\tau))\, x_\alpha(\tau)\, d\tau \tag{3.9}$$

where

$$\overline{\Phi}(z) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \frac{e^{\psi_k z}}{\lambda + \psi_k} \tag{3.10}$$

By combining the terms which are proportional to λ with the terms containing variable coefficients, one can obtain the following system of integral equations:

$$x_s(t) = \int_0^T \sum_{k=1}^{n} \sum_{\alpha=1}^{n} \Psi_{sk}(t-\tau)(m_{k\alpha} - \lambda\delta_{k\alpha})\, x_\alpha(\tau)\, d\tau \tag{3.11}$$

By taking the integral equations in the form (3.11), we obtain the expansion of Hill's determinant in powers of λ, and analogously to the cases (3.4) and (3.9), the coefficient $\Delta_i(\lambda)$ will be the sum of fractional functions of λ. The question on the choice of the matrix $\|B\|$ for the best convergence of the corresponding series will be the topic of a separate discussion.

As an example, let us consider the homogeneous system corresponding to (2.18) and the homogeneous integral equation with the kernel (2.20). Making use of the formulas derived above, we obtain

$$\alpha_k = \sum_{p_1=-\infty}^{\infty} \sum_{p_2=-\infty}^{\infty} \cdots \sum_{p_k=-\infty}^{\infty} \frac{M(\lambda+\psi_{p_1})}{D(\lambda+\psi_{p_1})} f_{p_1-p_2} \frac{M(\lambda+\psi_{p_2})}{D(\lambda+\psi_{p_2})} \cdots \frac{M(\lambda+\psi_{p_k})}{D(\lambda+\psi_{p_k})} f_{k_p-p_1} \tag{3.12}$$

where the f_i are the Fourier coefficients of the function $f(t)$.

Each of the terms in the expansion of $\Delta(\lambda, \mu)$ can be written in finite form.

From the general theory for the equation with the kernel $K(t, y)$ we have [4]

$$\Delta_n(\lambda) = \int_0^T \int_0^T \cdots \int_0^T \det \begin{vmatrix} K(t_1, t_1), \ldots, K(t_1, t_n) \\ K(t_n, t_1), \ldots, K(t_n, t_n) \end{vmatrix} dt_1, \ldots, dt_n \tag{3.13}$$

Let us consider the quantity (3.13) as a function of λ. In view of (2.20), this leads to the consideration of the determinant

$$K_n(\lambda) = \det \begin{vmatrix} \Psi(0) & \ldots & \Psi(t_1 - t_n) \\ \cdots & \cdots & \cdots \\ \Psi(t_n - t_1) & \ldots & \Psi(0) \end{vmatrix} \tag{3.14}$$

For the sake of simplicity, let us assume that the roots of Equation (1.3) are simple. Then one can show that the function $K_n(\lambda)$ has only simple poles at the points $\lambda = \lambda_0 + \psi_k$, $\rho = 1, \ldots, n$, where k is an arbitrary integer. Here, the residue of the function $K_n(\lambda)$ at the point $\lambda = \lambda_0 + \psi_k$ does not depend on k_n, and is equal to

$$K_{n\rho} = \frac{M(\lambda_\rho)}{D'(\lambda_\rho)} \sum_{s=1}^{n} K_{n\rho}{}^{s}$$

where $K_{n\rho}{}^{s}$ is the determinant obtained from (3.14) in the following way:

1) The sth column is replaced by a column consisting of ones;

2) In place of the functions $\Psi(t_i - t_j)$ one substitutes the functions

$$\Psi_\rho(t_i - t_j) = \sum_{k=-\infty}^{\infty}{}' \frac{M(\lambda_\rho + \psi_k)}{D(\lambda_\rho + \psi_k)} e^{\psi_k(t_i - t_j)}$$

where the prime indicates that the term corresponding to $k = 0$ has been omitted.

From what has been said, it follows that the expansion of the determinant $K_n(\lambda)$ into simple partial fractions has the form

$$K_n(\lambda) = \sum_{\rho=1}^{\infty} \sum_{k=-\infty}^{\infty} K_{n\rho} \frac{1}{\lambda - \lambda_\rho + \psi_k} = \frac{T}{2} \sum_{\rho=1}^{n} K_{n\rho} \coth \frac{(\lambda - \lambda_\rho) T}{2} \tag{3.15}$$

Substituting (3.15) into (3.13), we obtain

$$\Delta_k(\lambda) = \sum_{\rho=1}^{n} \coth \frac{(\lambda - \lambda_\rho) T}{2} \Delta_{k\rho} \tag{3.16}$$

where

$$\Delta_{k\rho} = \frac{T}{2} \int_0^T \cdots \int_0^T f(t_1) \ldots f(t_n) K_{n\rho}(t_1, \ldots, t_n)\, dt_1,\ dt_2, \ldots,\ dt_n \tag{3.17}$$

and, hence

$$\Delta(\lambda, \mu) = 1 + \sum_{\rho=1}^{n} \coth \frac{(\lambda - \lambda_\rho) T}{2} \Delta_\rho(\mu) \qquad \left(\Delta_\rho(\mu) = \sum_{n=1}^{\infty} (-1)^n \frac{\mu^n}{n!} \Delta_{n\rho} \right) \tag{3.18}$$

A result similar to (3.18) was obtained in [5]; there, however, the quantities $\Delta_\rho(\mu)$ were expressed in the form of infinite determinants.

Formulas (3.17) and (3.19) always yield convergent expansions of the infinite determinants $\Delta_\rho(\mu)$ in powers of the parameter, and they thus can serve as effective means for computing the determinants.

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Translated by H.P.T.